

**Exercice 1. Cyclic groups.** Let  $C_n$  be a cyclic group of order  $n$ .

1. Prove that  $\mathbb{Z}C_n$  is isomorphic to  $\mathbb{Z}[t]/(t^n - 1)$ .
2. Construct a free  $\mathbb{Z}C_n$ -resolution of  $\mathbb{Z}$ .
3. Compute  $H_k(C_n; \mathbb{Z})$  for all  $k \geq 0$ .

**Solution 1.**

1. Recall  $C_n = \langle s | s^n \rangle$ . Consider the ring morphism

$$\psi : \mathbb{Z}[t] \rightarrow \mathbb{Z}C_n : t \mapsto s$$

This map is surjective, as for every element  $\sum_{i=0}^{n-1} a_i s^i \in \mathbb{Z}C_n$ , we have that  $\psi(\sum_{i=0}^{n-1} a_i t^i) = \sum_{i=0}^{n-1} a_i s^i$ . Then,  $(t^n - 1) \subset \ker(\psi)$  as  $\psi(t^n - 1) = s^n - 1 = 0$ . The free  $\mathbb{Z}$ -modules  $\mathbb{Z}[t]/(t^n - 1)$  and  $\mathbb{Z}C_n$  have the same rank and the induced map  $\bar{\psi}$  is surjective, so we can conclude.

2. Define  $S : \mathbb{Z}C_n \xrightarrow{\cdot(t-1)} \mathbb{Z}C_n$  and  $N : \mathbb{Z}C_n \xrightarrow{\cdot(1+t+\dots+t^{n-1})} \mathbb{Z}C_n$ . Since  $(1+t+\dots+t^{n-1})(t-1) = 0$ ,  $S \circ N = N \circ S = 0$ . Thus, we have a chain complex :

$$\dots \xrightarrow{S} \mathbb{Z}C_n \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{S} \mathbb{Z}C_n \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where  $\varepsilon : \mathbb{Z}C_n \rightarrow \mathbb{Z}$  is the augmentation map.

**Exactness :**

- Since  $\varepsilon(1) = 1$  for example,  $\varepsilon$  is surjective.
  - $\text{Ker } \varepsilon = \text{Im } S$ . Let  $p = \sum_{i=0}^{n-1} a_i t^i \in \text{Ker}(\varepsilon)$ . According to the lecture notes,  $\text{Ker } \varepsilon$  is generated by  $t^n - 1$  as a free  $\mathbb{Z}$ -module. Now, it suffices to remark that  $t - 1$  divides  $t^n - 1$ , so that  $\text{Ker } \varepsilon \subset (t^n - 1)$ . The other inclusion follows easily from the lecture notes assertion (p. 11).
  - $\text{Im } N = \text{Ker } S$ . Let  $x = \sum_{i=0}^{n-1} a_i t^i \in \text{Ker}(S)$ . Then,  $\sum_{i=0}^{n-1} a_i t^i (t-1) = \sum_{i=0}^{n-1} (a_{i-1} - a_i) t^i = 0$ , where  $a_{-1} := 0$ . Then,  $a_i = a_{i-1} =: a$  so that  $x = a \cdot \sum_{i=0}^{n-1} a_i s^i \in \text{Im } N$ .
  - $\text{Im } S = \text{Ker } N$ . Let  $y = \sum_{i=0}^{n-1} b_i t^i \in \text{Ker}(N)$ . Then,  $(\sum_{j=0}^{n-1} b_j t^j)(\sum_{i=0}^{n-1} t^i) = 0$ , but since  $t^j \cdot (\sum_{i=0}^{n-1} t^i) = \sum_{i=0}^{n-1} t^i$ , we get  $(\sum_{j=0}^{n-1} b_j)(\sum_{i=0}^{n-1} t^i) = 0$ , hence  $\sum_{j=0}^{n-1} b_j = 0$ . We can then use again the small argument p.11 of the lecture notes to get that  $y \in (t - 1) = \text{Im } S$ .
3. We know from the lectures that given a free resolution of  $\mathbb{Z}$  as a  $\mathbb{Z}G$  module, we obtain  $H_\bullet(G; \mathbb{Z})$  via the homology of the complex of invariants. The complex of invariants is given as follows,

$$\dots \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}$$

since now  $g, \dots, g^{n-1}$  act trivially. Finally, we get

$$H_k(C_n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}/n\mathbb{Z}, & k \text{ odd}, k \geq 1, \\ 0, & k \text{ even}, k \geq 2. \end{cases}$$

**Exercise 2. Universal covers.** In the following cases describe a contractible CW-complex equipped with a free  $G$ -action on its cells, identify its cellular chain complex, and compute the homology groups  $H_k(G; \mathbb{Z})$  for all  $k \geq 0$ .

1.  $G = 1$ , the trivial group.
2.  $G = \mathbb{Z} \times \mathbb{Z}$ .
3.  $G = F(a, b)$ , free group on two generators.

**Solution 2.** We use the same technique for the three points of the exercise. Consider a group  $G$ , a **contractible** CW-complex  $X$ , and a group action of  $G$  on  $X$  which freely permutes the cells of  $X$  of each dimension. By definition of cellular homology, we have that  $C_n^{\text{cell}}(X; \mathbb{Z}) = \bigoplus_{n\text{-cells}} \mathbb{Z}$ . Moreover the chain complex

$$C_{\bullet}^{\text{cell}}(X; \mathbb{Z}) \xrightarrow{\epsilon} \mathbb{Z}$$

is a free resolution of  $\mathbb{Z}$  since  $X$  is contractible. The action of  $G$  on the cells of  $X$  turns  $\bigoplus_{n\text{-cells}} \mathbb{Z}$  into a  $\mathbb{Z}G$ -module. For each  $G$ -orbit  $\alpha$  of  $n$ -cells in  $X$ , fix a  $n$ -cell  $e_{\alpha}^n$  representing it, and consider the  $\mathbb{Z}G$ -module homomorphisms

$$f_{\alpha}^n: \mathbb{Z}G \rightarrow \bigoplus_{n\text{-cells}} \mathbb{Z}.$$

$$1 \mapsto 1 \cdot e_{\alpha}^n$$

Hence, there is a homomorphism of  $\mathbb{Z}G$ -modules

$$f^n: \bigoplus_{G\text{-orbits of } n\text{-cells}} \mathbb{Z}G \xrightarrow{\cong} \bigoplus_{n\text{-cells}} \mathbb{Z} = C_n^{\text{cell}}(X; \mathbb{Z})$$

which is surjective since each cell is in its own orbit, and which is injective since the action is free, so that for example  $f_{\alpha}^n(g) = g \cdot e_{\alpha}^n \neq e_{\alpha}^n$ . Hence  $C_{\bullet}^{\text{cell}}(X; \mathbb{Z}) \xrightarrow{\epsilon} \mathbb{Z}$  is a resolution of  $\mathbb{Z}$  by free  $\mathbb{Z}G$ -modules, so that from the lectures,

$$H_{\bullet}(G; \mathbb{Z}) \cong H_{\bullet}\left(\left(C_{\bullet}^{\text{cell}}(X; \mathbb{Z})\right)_G\right).$$

Considering the space  $X/G$ , which has a single cell for each orbit, we notice that  $C_n^{\text{cell}}(X/G; \mathbb{Z})$  must be the largest quotient of  $C_n^{\text{cell}}(X; \mathbb{Z})$  having a trivial action, namely its covariants. Thus,

$$H_{\bullet}(G; \mathbb{Z}) \cong H_{\bullet}(X/G; \mathbb{Z}).$$

So for the three questions we have to identify a certain contractible CW-complex and its quotient, and know the homology of the quotient.

1. First consider the CW-complex  $X = *$  on which  $G$  acts freely. Its cellular chain complex is

$$\cdots \rightarrow 0 \rightarrow \mathbb{Z}G = \mathbb{Z}$$

and the homology groups are

$$H_k(G; \mathbb{Z}) = H_k(X/G; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. Consider  $X = \mathbb{R}^2$  with  $G$ -action given by translation  $(a, b) \cdot (x, y) = (x + a, y + b)$ . Consider the cell structure given by  $X^0 = \mathbb{Z} \times \mathbb{Z}$ , the 1-cells are the vertical and horizontal unit intervals between the vertices, and the 2-cells are the full squares attached to the net  $X^1$ , so that there is a 2-cell for each pair of integers  $(a, b) \in \mathbb{Z}^2$ .  $G$  acts freely on the cells of  $X$  by integral translation. The cellular chain complex is

$$\cdots \rightarrow 0 \rightarrow \mathbb{Z}G \xrightarrow{d_2} (\mathbb{Z}G)^2 \xrightarrow{d_1} \mathbb{Z}G,$$

where

$$d_2: ABCD \mapsto \begin{pmatrix} AB - DC \\ BC - AD \end{pmatrix}$$

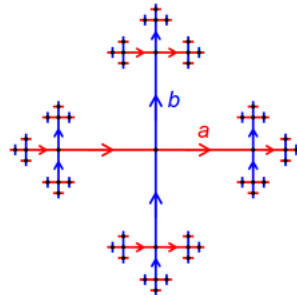
and

$$d_1: \begin{pmatrix} AB \\ CD \end{pmatrix} \mapsto B - A + D - C$$

We identified the first component of  $(\mathbb{Z}G)^2$  with the horizontal edges, and the second component with the vertical edges. The quotient space is  $X/G = S^1 \times S^1$  and the homology groups of  $G$  are

$$H_k(G; \mathbb{Z}) = H_k(T^1; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

3. We consider  $X = \Gamma(F(a, b))$  the Cayley graph of the free group  $F(a, b)$  on two generators. It has one vertex for each element of the group, and there is an edge between  $\omega_1, \omega_2 \in F(a, b)$  if and only if  $\omega_1 a = \omega_2$  or  $\omega_1 b = \omega_2$  for  $\{i, j\} = \{1, 2\}$ . It can be depicted as



The group  $F(a, b)$  acts on the vertices by left concatenation of words, and similarly on the edges, i.e.  $a \cdot (\omega_1, \omega_2) = (a\omega_1, a\omega_2)$  and as well for the action of  $b \in F(a, b)$ . In the above picture the action of  $a$  on an edge translate it to the right, and the action of  $b$  is an upper translation. It is clearly a free action. By construction  $X$  is contractible. There is a single 0-orbit and two orbits on the 1-cells with representatives  $(1, a)$  and  $(1, b)$ . Therefore  $X/G \cong S^1 \vee S^1$ . The cellular chain complex of  $X$  is

$$\cdots \rightarrow 0 \rightarrow (\mathbb{Z}G)^2 \xrightarrow{d_1} \mathbb{Z}G,$$

where  $d_1: \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto a - 1$  and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto b - 1$ . The homology groups of  $G$  are

$$H_k(G; \mathbb{Z}) \cong H_k(S^1 \vee S^1; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}^2 & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

◇ **Exercise 3. The bar resolution.** Let  $G$  be a group and  $F_\bullet \rightarrow \mathbb{Z}$  be the augmented bar resolution where  $F_n$  is the free  $\mathbb{Z}$ -module on  $G^{n+1}$ . We write

$$\partial_i(g_0, g_1, \dots, g_n) = (g_0, \dots, \hat{g}_i, \dots, g_n)$$

for the  $n$ -uple obtained by forgetting the  $i$ -th entry  $g_i$ , and construct then  $d: F_n \rightarrow F_{n-1}$  by setting  $d = \sum (-1)^i \partial_i$ .

1. Show that the augmented bar resolution is a chain complex.
2. Show that it is acyclic by constructing a contracting homotopy of  $\mathbb{Z}$ -modules (see Brown's book, page 18 for more hints).
3. Let  $G$  act by left multiplication on  $G^{n+1}$  componentwise and show that  $F_\bullet$  is a resolution of  $\mathbb{Z}$  by free  $\mathbb{Z}G$ -modules. We choose  $(1, g_1, g_1 g_2, \dots, g_1 \dots g_n)$  as representative for each free orbit of generators and call  $[g_1 | \dots | g_n]$  the class in the coinvariants.
4. Conclude that  $H_*(G; \mathbb{Z})$  is the homology of the coinvariants of  $F_\bullet$ .

**Solution 3.** (by Kasimir and Ángel)

1. To show that  $(F_\bullet)$  is a chain complex, we need to verify that  $dd = 0$ .

Since elements of the form  $(g_0, \dots, g_n)$ ,  $g_i \in G$  generate the free  $\mathbb{Z}$ -module  $F_n$  and  $d$  is linear, it is enough to show that  $dd = 0$  on such elements to prove that  $dd = 0$  in general.

We introduce the following notation : let  $i \in \{0, \dots, n\}$ ,  $j \in \{0, \dots, n-1\}$ , and  $(g_0, \dots, g_n) \in F_n$  :

$$\begin{aligned} \text{if } i > j : \alpha_{ij} &= (-1)^{i+j} (g_0, \dots, \hat{g}_j, \dots, \hat{g}_i, \dots, g_n) \\ \text{if } i \leq j : \alpha_{ij} &= (-1)^{i+j} (g_0, \dots, \hat{g}_i, \dots, \hat{g}_{j+1}, \dots, g_n) \end{aligned}$$

This notation is useful because it allows us to write double composition of the maps  $\partial_i$  in a compact way :

$$\text{if } i > j : \partial_j \partial_i (g_0, \dots, g_n) = \alpha_{ij} \tag{1}$$

$$\text{if } i \leq j : \partial_j \partial_i (g_0, \dots, g_n) = \alpha_{ij} \tag{2}$$

(2) holds because the position of every element of the vector  $(g_0, \dots, \hat{g}_i, \dots, g_n)$  after  $\hat{g}_i$  has their position shifted by one backwards.

Let  $i \leq j$ . The following holds :

$$\alpha_{ij} = (-1)^{i+j} (g_0, \dots, \hat{g}_i, \dots, \hat{g}_{j+1}, \dots, g_n) = -\alpha_{(j+1)i}$$

Hence,

$$dd(g_0, \dots, g_n) = \sum_{j=0}^{n-1} \sum_{i=0}^n \alpha_{ij} \tag{3}$$

$$= \sum_{i \leq j} \alpha_{ij} + \alpha_{(j+1)i} \tag{4}$$

$$= \sum_{i \leq j} -\alpha_{(j+1)i} + \alpha_{(j+1)i} = 0 \tag{5}$$

In (4), we take the indices  $i \in \{0, \dots, n\}$  and  $j \in \{0, \dots, n-1\}$ . Let us quickly check that we indeed have all the terms of the sum in the line (4). If  $i-1 \geq j$  then  $\alpha_{ij}$  appears in the term  $(j-1, i)$  of the sum, namely  $\alpha_{j(i-1)} + \alpha_{ij}$ . Our counting in (3) reaches all instances of  $\alpha_{ij}$ . There is no double counting : our summation has exactly  $n(n+1)$  elements. Indeed, for each  $j$ , we have  $2(j+1)$  pairs  $(ij)$  that give the  $\alpha$ . They indeed sum up to  $n(n+1)$  :

$$\sum_{j=0}^{n-1} 2j = n(n+1)$$

2. To construct a contracting homotopy for the chain  $F_\bullet \rightarrow \mathbb{Z}$  means that we want  $\forall n \in \mathbb{Z}_{\geq 0}$  a map  $h_n : F_n \rightarrow F_{n+1}$  such that

$$Id_{F_n} - 0_{F_n} = d_{n+1}h_n + h_{n-1}d_n$$

( $0_{F_n}$  being the zero morphism)

Note that this immediately implies that our chain complex is acyclic. Indeed, homotopic maps induce the same map in homology. As the morphisms  $Id_{F_n}$  and  $0_{F_n}$  induce respectively  $Id_{H_n(F_\bullet)}$  and  $0_{H_n(F_\bullet)}$ , we have that  $Id_{H_n(F_\bullet)} = 0_{H_n(F_\bullet)}$ , so  $H_n(F_\bullet) = 0, \forall n$ .

We define the map  $h_n$  as

$$h_n : \begin{cases} F_n \rightarrow F_{n+1} : (g_0, \dots, g_n) \mapsto (1, g_0, \dots, g_n) & \text{for } n \geq 0 \\ h_{-1} : \mathbb{Z} \rightarrow F_0 : 1 \mapsto 1 & \text{for } n = -1 \\ 0 & \text{for } n = -2 \end{cases}$$

( $h_1$  is the map induced by mapping  $1 \in \mathbb{Z}$  to  $1 \in G$ ). Let's verify it satisfies the equation above :

For  $n \geq 0$ , we have that  $\forall (g_0, \dots, g_n) \in F_n$ ,

$$\begin{aligned} (d_{n+1}h_n + h_{n-1}d_n)(g_0, \dots, g_n) &= d_{n+1}(1, g_0, \dots, g_n) + h_{n-1}\left(\sum_{i=0}^n (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_n)\right) \\ &= \sum_{i=-1}^n (-1)^{i+1} (g_{-1}, g_0, \dots, \hat{g}_i, \dots, g_n) + \left(\sum_{i=0}^n (-1)^i (g_{-1}, g_0, \dots, \hat{g}_i, \dots, g_n)\right) \\ &= (\hat{g}_{-1}, g_0, \dots, g_n) - \left(\sum_{i=0}^n (-1)^i (g_{-1}, g_0, \dots, \hat{g}_i, \dots, g_n)\right) \\ &\quad + \left(\sum_{i=0}^n (-1)^i (g_{-1}, g_0, \dots, \hat{g}_i, \dots, g_n)\right) = (g_0, \dots, g_n) \end{aligned}$$

where  $g_{-1} = 1 \in G$ . The key point here is that the first sum puts the hat on  $g_{-1}$ , while the second sum doesn't.

For  $n = -1$ , we have that for  $z \in \mathbb{Z}$ ,

$$(d_0h_{-1} + h_{-2}d_{-1})(z) = (d_0h_{-1} + 0)(z) = d_0(z \cdot 1) + 0 = z$$

So indeed,  $h_n$  defines a contracting homotopy.

3. We already know that  $F_\bullet \rightarrow \mathbb{Z}$  is a  $\mathbb{Z}$ -module resolution. Also, the  $F_n$  are clearly free  $\mathbb{Z}G$ -modules when we define their scalar multiplication using the  $G$  action given in the statement. Therefore, we just need to check that the differential maps  $d_n$  and the augmentation map  $\epsilon$  commute with the scalar multiplication by  $G$ .

For the differential maps, we have  $\forall n, (g_0, \dots, g_n)^1 \in F_n, g \in G$  that

$$\begin{aligned} & d(g \cdot (g_0, \dots, g_n)) \\ &= d(gg_0, \dots, gg_n) = \sum_{i=0}^n (-1)^i (gg_0, \dots, \widehat{gg_i}, \dots, gg_n) = g \cdot \left( \sum_{i=0}^n (-1)^i (g_0, \dots, \widehat{g_i}, \dots, g_n) \right) \\ &= g \cdot d(g_0, \dots, g_n) \end{aligned}$$

And for the augmentation map, we have  $\forall (g_0) \in F_0, g \in G$  that

$$\epsilon(g \cdot (g_0)) = 1 = g \cdot 1 = g \cdot \epsilon((g_0))$$

4. Applying the functor  $- \otimes_{\mathbb{Z}G} \mathbb{Z}$  to  $F_\bullet$  yields a chain complex, the homology groups of which are precisely the homology groups  $H_n(G, \mathbb{Z})$ , because  $F_\bullet$  is a free  $\mathbb{Z}G$ -resolution of  $\mathbb{Z}$ , and  $H_\bullet(G, \mathbb{Z})$  is defined as the left derived functor to  $- \otimes_{\mathbb{Z}G} \mathbb{Z}$ .

However, as seen in the lecture notes, the group of coinvariants of a  $G$ -module  $M$  is isomorphic to  $M \otimes_{\mathbb{Z}G} \mathbb{Z}$ . We actually have a stronger result : this is a natural isomorphism. This tells us that the chain complexes  $(F_\bullet)_G$  and  $(F_\bullet) \otimes_{\mathbb{Z}G} \mathbb{Z}$  are isomorphic, so they yield the same homology.

**Remark :** We didn't actually see in the notes the naturality of the isomorphism  $M_G \cong M \otimes_{\mathbb{Z}G} \mathbb{Z}$ , but the proof is similar to the strategy we used to prove the isomorphism on the objects.

Because the point of the exercise was to understand that the bar resolution gives us a free resolution  $\mathbb{Z}G$  resolution of  $\mathbb{Z}$  that can be used to compute the homology groups  $H_k(G; \mathbb{Z})$ , we didn't see fit to include the proof of the naturality.

◇ **Exercice 4. The bar resolution, computations.** Let  $G$  be a group and  $F_\bullet \rightarrow \mathbb{Z}$  be the augmented bar resolution.

1. Identify  $F_2, F_1$  and  $F_0$ , as well as the differentials  $d_1$  and  $d_2$ .
2. Identify the coinvariants  $(F_2)_G, (F_1)_G$ , and  $(F_0)_G$ , as well as the induced differentials in terms of the bar notation defined in the previous exercise.
3. Compute  $H_0(G; \mathbb{Z})$  and  $H_1(G; \mathbb{Z})$ . (*Hint.* Define a map  $(F_1)_G \rightarrow G_{ab}$  by sending  $[g]$  to the class  $g[G, G]$ .)

**Solution 4.** (by Boran and Zsigmond)

1. By definition  $F_0 = \mathbb{Z}[G], F_1 = \mathbb{Z}[G^2], F_2 = \mathbb{Z}[G^3]$  as Abelian groups and  $G$  acts on  $F_\bullet$  coordinate-wise. The differentials are defined as :

$$\begin{aligned} d_1: \mathbb{Z}[G^2] &\longrightarrow \mathbb{Z}[G], & \sum_{g_1, g_2 \in G} n_{g_1, g_2} (g_1, g_2) &\longmapsto \sum_{g_1, g_2 \in G} n_{g_1, g_2} (g_2 - g_1), \\ d_2: \mathbb{Z}[G^3] &\longrightarrow \mathbb{Z}[G^2], & \sum_{g_1, g_2, g_3 \in G} n_{g_1, g_2, g_3} (g_1, g_2, g_3) &\longmapsto \sum_{g_1, g_2, g_3 \in G} n_{g_1, g_2, g_3} ((g_2, g_3) - (g_1, g_3) + (g_1, g_2)). \end{aligned}$$

1. a priori, we should verify the property for formal sums in  $F_n$ . However, it suffices to do so for elements of the form  $(g_0, \dots, g_n)$ , as we can conclude the general case using that the  $d_n$  are  $\mathbb{Z}$ -module morphisms. This remarks applies for the augmentation map as well.

2. The modules  $(F_0)_G$ ,  $(F_1)_G$ ,  $(F_2)_G$  are free  $\mathbb{Z}$ -modules and  $G$  acts on them trivially. The modules are generated by  $[\ ]$ ,  $[g_1]$ ,  $[g_1 \mid g_2]$  respectively, where  $g_1, g_2 \in G$ . Since  $d_1([g]) = d_1(1, g) = (1) - (g) = [\ ] - [g] = [\ ] - [\ ] = 0$ , the map  $(d_1)_G$  sends every basis element from  $(F_1)_G$  to zero. Therefore, it is the zero-map.

$$d_2[g_1 \mid g_2] = d_2(1, g_1, g_1g_2) = (g_1, g_1g_2) - (1, g_1g_2) + (1, g_1), \text{ therefore,}$$

$$(d_2)_G([g_1 \mid g_2]) = g_1[g_2] - [g_1g_2] + [g_1] = [g_2] - [g_1g_2] + [g_1].$$

where we used that  $g_1$  acts trivially.

3. Since  $(d_1)_G$  is the zero map,  $H_0(G; \mathbb{Z}) = (F_0)_G = \mathbb{Z}$ .

We have that  $H_1(G; \mathbb{Z}) = (F_1)_G / \text{Im}((d_2)_G)$ .

Consider the map  $\phi : G \rightarrow H_1(G; \mathbb{Z})$  defined by  $g \mapsto [g]$ . (To be pedantic, it maps to the coset of  $[g]$  in  $H_1(G; \mathbb{Z})$ .)

Let's first show that this is a group homomorphism. Let  $g, h \in G$ . Then,  $\phi(gh) = [gh] = [g] + [h] = \phi(g) + \phi(h)$ . This follows by the computation in part (b) by taking  $[g_1 \mid g_2] = [g \mid h]$ . Notice that  $H_1(G; \mathbb{Z})$  is an Abelian group since it is the quotient of an Abelian group, so by the universal property of the Abelianization we get a map  $\phi : G_{ab} \rightarrow H_1(G; \mathbb{Z})$ . (We're calling both maps  $\phi$  by a minor abuse of notation.)

Now, consider the map  $\psi : (F_1)_G \rightarrow G_{ab}$  defined by using the universal property of free Abelian groups and the set-map  $[g] \mapsto g$ . This induces a unique group homomorphism out of  $(F_1)_G$ .

Let  $g, h \in G$ . Then, we have that  $-[gh] \mapsto (gh)^{-1} = h^{-1}g^{-1}$ , so  $[g] + [h] - [gh] \in \ker(\psi)$ . Thus, we get an induced map (which we again call  $\psi$ )  $\psi : H_1(G; \mathbb{Z}) \rightarrow G_{ab}$  sending  $[g] \mapsto g$ .

To finish the proof, just notice that both compositions of  $\phi$  and  $\psi$  give the identity map.

◇ **Exercise 5. Hochschild complex.** Let  $k$  be a commutative ring,  $A$  a (unital)  $k$ -algebra with multiplication  $\mu$ , and  $M$  an  $A$ -bimodule (the left and right actions commute :  $a(mb) = (am)b$ ). We write  $\otimes$  for the tensor product of  $k$ -modules.

Define  $d_i : M \otimes A^{\otimes n} \rightarrow M \otimes A^{\otimes(n-1)}$  for  $0, 1 \leq i \leq n-1$ , and  $n$  by the following formulas :

$$\begin{aligned} d_0(m \otimes a_1 \otimes \cdots \otimes a_n) &= ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ d_i(m \otimes a_1 \otimes \cdots \otimes a_n) &= m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ d_n(m \otimes a_1 \otimes \cdots \otimes a_n) &= a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

The *Hochschild boundary* is the  $k$ -linear map  $b = \sum (-1)^i d_i$ .

1. Prove that  $b \circ b = 0$ . *Hint.* Prove that  $d_i d_j = d_{j-1} d_i$  when  $i < j$ .
2. Write  $C_\bullet(A; M)$  for the *Hochschild complex* and  $C_\bullet(A) = C_\bullet(A; A)$ . Show that this construction is functorial in  $M$  and  $A$  (define the bimodule structure you need here).
3. We define  $HH_*(A) = H_*(C_\bullet(A))$ . Compute  $HH_0(A; M)$ ,  $HH_0(A)$ , and  $HH_n(k)$  for all  $n \geq 0$ .

**Solution 5** (by Elina and Thomas)

1. Let us first show that  $d_i d_j = d_{j-1} d_i$  if  $i < j$ . Showing this fact for the generators is sufficient using linearity. The key observation is that, since  $i$  is smaller than  $j$ , applying  $d_i$  first shifts the indices of all tensor elements of index greater or equal to  $i$ , including that of  $a_j$ .

We consider two cases. Suppose  $i = j - 1$ .

- $j = 1$  : then  $i = 0$ . Consequently, we get

$$d_0 d_1 (m \otimes a_1 \otimes \dots \otimes a_n) = d_0 (m \otimes a_1 a_2 \otimes a_3 \otimes \dots \otimes a_n) = m a_1 a_2 \otimes a_3 \otimes \dots \otimes a_n$$

and

$$d_0 d_0 (m \otimes a_1 \otimes \dots \otimes a_n) = d_0 (m a_1 \otimes a_2 \otimes a_3 \otimes \dots \otimes a_n) = m a_1 a_2 \otimes a_3 \otimes \dots \otimes a_n.$$

Thus, we see clearly that  $d_0 d_1 = d_0 d_0$ .

- $j = n$  : then  $i = n - 1$ . This time, we get

$$d_{n-1} d_n (m \otimes a_1 \otimes \dots \otimes a_n) = d_{n-1} (a_n m \otimes a_1 \otimes \dots \otimes a_{n-1}) = a_{n-1} a_n m \otimes a_1 \otimes \dots \otimes a_{n-2}$$

and

$$d_{n-1} d_{n-1} (m \otimes a_1 \otimes \dots \otimes a_n) = d_{n-1} (m \otimes a_1 \otimes \dots \otimes a_{n-1} a_n) = a_{n-1} a_n m \otimes a_1 \otimes \dots \otimes a_{n-2}.$$

Thus, both coincide again.

- $1 < j < n$  : In that case, we get

$$\begin{aligned} d_{j-1} d_j (m \otimes a_1 \otimes \dots \otimes a_n) &= d_{j-1} (m \otimes a_1 \otimes \dots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes \dots \otimes a_n) \\ &= m \otimes a_1 \otimes \dots \otimes a_{j-1} a_j a_{j+1} \otimes \dots \otimes a_n \\ &= d_{j-1} (m \otimes a_1 \otimes \dots \otimes a_{j-1} a_j \otimes a_{j+1} \otimes \dots \otimes a_n) \\ &= d_{j-1} d_{j-1} (m \otimes a_1 \otimes \dots \otimes a_n) \end{aligned}$$

If  $i \neq j - 1$ , one may once again separate such specific cases, dealing separately with  $i = 0$ ,  $0 < i < j < n$  and  $j = n$ . The computations are similar, except this time we do not get double multiplications (e.g.  $a_{j-1} a_j a_{j+1}$ ).

Combining all that, we have successfully proved that  $d_i d_j = d_{j-1} d_i$  whenever  $i < j$ . Then, the argument is the same as Exercise 3.1. Namely, the sum

$$d \circ d = \sum (-1)^{i+j} d_i d_j, \quad 0 \leq j \leq n, \quad 0 \leq i \leq n - 1$$

splits into two parts according to  $i < j$  or  $i \geq j$ . The term  $(-1)^{i+j} d_i d_j$  of the first part cancels with the term  $(-1)^{j-1+i} d_{j-1} d_i$  of the second part.

**Remark :** This common feature of the bar resolution and the Hochschild complex (and simplicial chain complex) has a name. A *presimplicial module*  $C$  is a collection of modules  $C_n$ ,  $n \geq 0$ , together with maps, called *face maps* or *face operators*,

$$d_i : C_n \rightarrow C_{n-1}, \quad i = 0, \dots, n$$

such that

$$d_i d_j = d_{j-1} d_i, \quad 0 \leq i < j \leq n.$$

Then, the argument above shows that  $d = \sum (-1)^i d_i$  is a chain map.

2. (For this question, Elina and Thomas wrote a much more detailed answer, and here is a summary) A bimodule homomorphism  $f : M \rightarrow M'$  induces a map

$$f_* : H_*(A, M) \rightarrow H_*(A, M'), \quad f_*(m, a_1, \dots, a_n) = (f(m), a_1, \dots, a_n).$$

The construction is also functorial in  $A$  in the following sense. Let  $g : A \rightarrow A'$  be a  $k$ -algebra map and  $M'$  be an  $A'$ -bimodule. Via  $g$  the module  $M'$  can be considered as an  $A$ -bimodule and there is defined a map

$$g_* : H_*(A, M') \rightarrow H_*(A', M')$$

given by

$$g_*(m, a_1, \dots, a_n) = (m, g(a_1), \dots, g(a_n)).$$

3. (Again, a summary) The group

$$H_0(A, M) = M_A = M/\{am - ma \mid a \in A, m \in M\}$$

is also called the *module of coinvariants* of  $M$  by  $A$ . Let  $[A, A]$  denote the additive commutator sub- $k$ -module generated by

$$[a, a'] = aa' - a'a, \quad a, a' \in A.$$

Then

$$HH_0(A) = A/[A, A].$$

If  $A$  is commutative, then  $HH_0(A) = A$ .

When  $A = k$ , the Hochschild complex for  $M = k$  is

$$\dots \longrightarrow k \otimes k \otimes k \longrightarrow k \otimes k \longrightarrow k,$$

therefore

$$HH_0(k) = k \quad \text{and} \quad HH_n(k) = 0 \quad \text{for } n > 0.$$

◇ **Exercise 6. Morita invariance for  $HH_0$ .** Denote by  $M_r(R)$  the associative ring of  $r \times r$ -matrices with entries in the ring  $R$ . With the same notation as in Exercise 5 we want to prove that the trace map  $M_r(R) \rightarrow R$  induces an isomorphism on  $HH_0$  for any ring  $R$ . We call  $Tr$  the composition of the trace map, followed by abelianization  $R \rightarrow R/[R, R]$ .

1. Explain which ring  $k$  and  $k$ -algebra  $A$  of exercise 5 we use here.
2. Show that  $[M_r(R), M_r(R)]$  is contained in the kernel of  $Tr$ .
3. Show that this kernel is generated by the elementary matrices  $E_{ij}(r)$  with  $i \neq j$  and diagonal matrices whose trace is in  $[R, R]$ .
4. Show that these generators all belong to  $[M_r(R), M_r(R)]$ .
5. Conclude that the trace map induces an isomorphism  $HH_0(M_r(R)) \cong HH_0(R)$ .

**Solution 6** (by Sandro and Patrick)

1. In Exercise 5 we proved that for a  $k$ -algebra  $A$  one has

$$HH_0(A) \cong A/[A, A],$$

where  $[A, A]$  denotes the  $k$ -submodule generated by commutators. To apply this to the matrix ring  $M_r(R)$  we must view both  $R$  and  $M_r(R)$  as  $k$ -algebras for some commutative ring  $k$ .

A canonical choice is  $k = Z(R)$ : every associative unital ring  $R$  is naturally a algebra over its center. Naturally  $M_r(R)$  is also a  $Z(R)$ -algebra, by considering the ring map  $z \in Z(R) \mapsto z \cdot \text{Id}$ .

With this choice we obtain

$$HH_0(R) \cong R/[R, R], \quad HH_0(M_r(R)) \cong M_r(R)/[M_r(R), M_r(R)].$$

2. Let  $A = (A_{ij}), B = (B_{ij}) \in M_r(R)$ . We want to show that  $\text{Tr}([A, B]) = 0$ , where  $[A, B] = AB - BA$ . Since  $\text{Tr}$  is a  $k$ -module map, this will imply that every  $k$ -linear combination of commutator in  $M_r(R)$  lies in the kernel of  $\text{Tr}$ , i.e.

$$[M_r(R), M_r(R)] \subseteq \ker(\text{Tr}).$$

First compute the diagonal entries of  $[A, B]$

$$([A, B])_{ii} = \sum_{k=1}^r A_{ik}B_{ki} - \sum_{k=1}^r B_{ik}A_{ki}.$$

Hence the trace is

$$\begin{aligned} \text{Tr}([A, B]) &= \sum_{i=1}^r ([A, B])_{ii} \\ &= \sum_{i=1}^r \sum_{k=1}^r A_{ik}B_{ki} - \sum_{i=1}^r \sum_{k=1}^r B_{ik}A_{ki}. \end{aligned}$$

In the second double sum, rename the indices  $(i, k) \mapsto (k, i)$  to get

$$\text{Tr}([A, B]) = \sum_{i,k=1}^n A_{ik}B_{ki} - \sum_{i,k=1}^n B_{ki}A_{ik} = 0 \quad \text{in } R/[R, R].$$

Thus the image of every commutator  $[A, B]$  under  $\text{Tr}$  is zero, and therefore

$$[M_r(R), M_r(R)] \subseteq \ker(\text{Tr}).$$

3. For  $a \in R$  and  $i \neq j$  we denote by  $E_{ij}(a) \in M_r(R)$  the matrix whose only nonzero entry is  $a$  in the  $(i, j)$ -position. These matrices generate all matrices supported off the diagonal.

Let  $M = (M_{ij}) \in M_r(R)$  satisfy  $\text{Tr}(M) \in [R, R]$ , i.e.  $\sum_{i=1}^r M_{ii} \in [R, R]$ . Write

$$M = \underbrace{\sum_{i \neq j} M_{ij} E_{ij}(1)}_{\text{off-diagonal part}} + \underbrace{\text{diag}(M_{11}, \dots, M_{rr})}_{\text{diagonal part}}.$$

The first summand is a linear combination of the elementary matrices  $E_{ij}(M_{ij}) = M_{ij}E_{ij}(1)$  with  $i \neq j$ . This summand does not contribute to the trace of  $M$ . The second summand is a diagonal matrix whose trace equals  $\text{Tr}(M)$ , and therefore lies in the set of diagonal matrices whose trace is in  $[R, R]$ .

Thus every  $M \in \ker \text{Tr}$  is a sum of a linear combination of the off-diagonal elementary matrices and a diagonal matrix with trace in  $[R, R]$ .

For the reverse inclusion observe that each generator belongs to  $\ker \text{Tr}$ : every  $E_{ij}(a)$  with  $i \neq j$  has zero trace, and any diagonal matrix whose trace lies in  $[R, R]$  is sent to zero in the abelianization  $R/[R, R]$ . Therefore the subgroup generated by these elements is contained in  $\ker \text{Tr}$ .

4. We first show that the off-diagonal elementary matrices are commutators. Let  $i \neq j$  and  $x \in R$ . Then

$$[E_{ij}(x), E_{jj}(1)] = E_{ij}(x)E_{jj}(1) - E_{jj}(1)E_{ij}(x) = E_{ij}(x).$$

It thus remains to show that diagonal matrices whose trace is in  $[R, R]$  are in  $[M_r(R), M_r(R)]$ . Consider  $x, y \in R$ . Then,

$$[E_{ij}(x), yE_{ji}(y)] = E_{ii}(xy) - E_{jj}(yx). \quad (6)$$

Hence, if  $D$  is a diagonal matrix whose trace is in  $[R, R]$ , its diagonal entries are either in  $[R, R]$  or they appear pairwise with different indices as in Equation (6). In particular, each of these cases can be solved by a commutator as in Equation (6). Hence, every diagonal matrix whose trace is in  $[R, R]$  is in  $[M_r(R), M_r(R)]$ .

5.  $\text{Tr}$  is a group homomorphism, which is clearly surjective, since for each  $r \in R$ ,  $\text{diag}(r, 0, \dots, 0)$  is mapped to the equivalence class of  $r$  in  $R/[R, R]$ . By the previous points, we further deduce that the kernel of  $\text{Tr}$  is precisely  $[M_r(R), M_r(R)]$ . Thus, we yield by the first isomorphism theorem an isomorphism

$$M_r(R)/[M_r(R), M_r(R)] \xrightarrow{\text{Tr}} R/[R, R].$$

As mentioned in the first part of the exercise, we have for a  $k$ -algebra  $A$

$$HH_0(A) \cong A/[A, A].$$

Hence,  $\text{Tr}$  induces an isomorphism

$$HH_0(M_r(R)) \cong HH_0(R).$$